

## Lecture 20

April 29<sup>th</sup>, 2004

### Difference Quotients and Sobolev spaces

Define

$$\Delta_i^h u := \frac{u(x + h \cdot \mathbf{e}_i) - u(x)}{h}, \quad h \neq 0.$$

**Lemma.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ , and  $u \in W^{1,p}(\Omega)$ , for some  $1 \leq p < \infty$ . Then for any  $\Omega' \subseteq \Omega$  such that  $\text{dist}(\Omega', \partial\Omega) > h$  holds

$$\|\Delta_i^h u\|_{L^p(\Omega')} \leq \|D_i u\|_{L^p(\Omega)}.$$

*Proof.*

$$\begin{aligned} |\Delta_i^h u| &= \left| \frac{u(x + h \cdot \mathbf{e}_i) - u(x)}{h} \right| \leq \frac{1}{h} \int_0^h |\mathbf{D}_i u(x_1, \dots, x_i + \zeta, \dots, x_n)| d\zeta \\ &\leq \frac{1}{h} \left\{ \int_0^h 1^q \right\}^{\frac{1}{q}} \left\{ \int_0^h |\mathbf{D}_i u(x_1, \dots, x_i + \zeta, \dots, x_n)|^p d\zeta \right\}^{\frac{1}{p}} \Rightarrow \\ |\Delta_i^h u|^p &\leq h^{\frac{p}{q}-p} \cdot \int_0^h |\mathbf{D}_i u(x_1, \dots, x_i + \zeta, \dots, x_n)|^p d\zeta \\ &= \frac{1}{h} \cdot \int_0^h |\mathbf{D}_i u(x_1, \dots, x_i + \zeta, \dots, x_n)|^p d\zeta \Rightarrow \\ \int_{\Omega'} |\Delta_i^h u|^p &\leq \frac{1}{h} \cdot \int_{\Omega'} \int_0^h |\mathbf{D}_i u|^p d\zeta d\mathbf{x} = \frac{1}{h} \cdot \int_0^h \int_{\Omega'} |\mathbf{D}_i u|^p d\mathbf{x} d\zeta \\ &= \frac{1}{h} \int_0^h \|\mathbf{D}_i u\|_{L^p(\Omega')}^p d\zeta = \|\mathbf{D}_i u\|_{L^p(\Omega')}^p \leq \|\mathbf{D}_i u\|_{L^p(\Omega)}^p, \end{aligned}$$

where we applied Fubini's Theorem in order to switch order of integration. ■

Conversely we have

**Lemma.** Let  $u \in L^p(\Omega)$  for some  $1 \leq p < \infty$  and suppose  $\Delta_i^h u \in L^p(\Omega')$  with  $\|\Delta_i^h u\|_{L^p(\Omega')} \leq K$  for all  $\Omega' \subseteq \Omega$  and  $0 < |h| < \text{dist}(\Omega', \Omega)$ . Then the weak derivative satisfies  $\|D_i u\|_{L^p(\Omega)} \leq K$ . Consequently if this holds for all  $i = 1, \dots, n$  then  $u \in W^{1,p}(\Omega)$ .

*Proof.* We will make use of

**Aloulou's Theorem.** A bounded sequence in a separable, reflexive Banach space has a weakly convergent subsequence.

A topological space is called *separable* if it contains a countable dense set.

A Banach space is called *reflexive* if  $(B^*)^* = B$ .

A sequence  $\{x_n\}$  in a Banach space is said to *converge weakly* to  $x$  when  $\lim_{n \rightarrow \infty} F(x_n) \rightarrow F(x)$  for all linear functionals  $F \in B^*$ . This is sometimes denoted  $\lim_{n \rightarrow \infty} x_n \rightharpoonup x$ .

**Example:** Let  $\ell^2 := \left\{ (a_1, a_2, \dots) : \sum_{i=1}^{\infty} a_i^2 < \infty \right\}$ . Consider the sequence  $\{x_i := (0, \dots, 0, 1, 0, \dots)\}$

$\subseteq \ell^2$ . Any bounded linear functional on  $\ell^2$  will be some linear combination of the linear functionals  $F_j$ , defined by  $F_j(a_1, \dots) = a_j$  (each such linear combination corresponds exactly to a point in  $\ell^2$ ).

That makes sense, indeed by the Riesz Representation Theorem  $(\ell^2)^* = \ell^2$  (note  $\ell^2$  is a Hilbert space not just a Banach space as it has an inner product structure). For any such  $F = (a_1, \dots)$ ,

$\lim_{i \rightarrow \infty} F(x_i) = \lim_{i \rightarrow \infty} a_i = 0$ . So  $x_i$  converges to the 0 vector weakly, though certainly not strongly:

by Fourier Theory each point in  $\ell^2$  corresponds to a periodic function on  $[0, 1]$ , i.e an element of  $L^2(S^1)$ , and of course  $\lim_{n \rightarrow \infty} \exp(n2\pi\sqrt{-1}z) \not\rightarrow 0(z)$ .

We come back to the proof. For the Banach space  $B = L^p(\Omega)$ ,  $B^* = L^q(\Omega)$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .

This can be seen directly: If  $F \in (L^p(\Omega))^*$ , then exists  $f$  such that  $F(g) = \int_{\Omega} g \cdot f$ ,  $\forall g \in L^p(\Omega)$ , and this will be bounded iff  $f \in L^q(\Omega)$ . So we get an identification  $F \in (L^p(\Omega))^* \cong L^q(\Omega)$ .

By Alouglou's Theorem there exists a sequence  $\{h_m\} \rightarrow 0$  with  $\Delta_i^{h_m} u \rightharpoonup v \in L^p(\Omega)$ . In other words

$$\int_{\Omega} \psi \cdot \Delta_i^{h_m} u \rightarrow \int_{\Omega} \psi \cdot v \in L^p(\Omega), \quad \forall \psi \in L^q(\Omega).$$

And in particular for any  $\psi \in \mathcal{C}_0^1(\Omega)$  (which is dense in  $L^q(\Omega)$  so will suffice to look at such  $\psi$  as will become clear ahead)

$$\begin{aligned} \int_{\Omega} \psi \Delta_i^{h_m} u &= \int_{\Omega} \psi \frac{1}{h} (u(x + h \cdot \mathbf{e}_1) - u(x)) d\mathbf{x} \\ &= \frac{1}{h} \int_{\Omega} \psi(x - h \mathbf{e}_1) u(x) d\mathbf{x} - \frac{1}{h} \int_{\Omega} \psi(x) u(x) d\mathbf{x} \\ &= \int_{\Omega} \frac{1}{h} (\psi(x - h \mathbf{e}_1) - \psi(x)) u(x) d\mathbf{x} \\ &= \int_{\Omega} -\Delta_i^h \psi(x) u(x) d\mathbf{x} \xrightarrow{h \rightarrow 0} \int_{\Omega} -D_i \psi(x) u(x) d\mathbf{x} \end{aligned}$$

since  $\psi$  is continuously differentiable. Altogether

$$\int_{\Omega} \psi \cdot v \in L^p(\Omega) = \int_{\Omega} -D_i \psi(x) u(x) d\mathbf{x},$$

which by definition means  $v$  is the weak derivative of  $u$  in the direction of the  $x_i$  axis, or simply the undistinctive notation  $v = D_i u$ .

We also get the desired estimate, using the Fatou Lemma  $\int \liminf \leq \liminf \int$

$$\int_{\Omega} |D_i u|^p d\mathbf{x} = \int_{\Omega} \liminf |\Delta_i^h u|^p d\mathbf{x} \leq \liminf \int_{\Omega} |\Delta_i^h u|^p d\mathbf{x} \leq K^p,$$

i.e  $\|D_i u\|_{L^p(\Omega)} \leq K$ . ■

## L<sup>2</sup> Theory

Consider the second order equation in divergence form

$$Lu \equiv L(u) := D_i(a^{ij}D_j u) + b^i D_i u + c \cdot u = f,$$

with  $a^{ij}, b^i, c \in L^1(\Omega)$  (integrable coefficients).

We call  $u \in W^{1,2}(\Omega)$  a *weak solution* of the equation if

$$\forall v \in \mathcal{C}_0^1(\Omega) \quad - \int_{\Omega} a^{ij} D_j u D_i v + \int_{\Omega} (b^i D_i u + cu)v = \int_{\Omega} f v.$$

## Elliptic Regularity

Let  $u \in W^{1,2}(\Omega)$  be a weak solution of  $Lu = f$  in  $\Omega$ , and assume

- $L$  strictly elliptic with  $(a^{ij}) > \gamma \cdot I$ ,  $\gamma > 0$
- $a^{ij} \in \mathcal{C}^{0,1}(\Omega)$
- $b^i, c \in L^\infty(\Omega)$
- $f \in L^2(\Omega)$

Then for any  $\Omega' \Subset \Omega$ ,  $u \in W^{2,2}(\Omega')$  and

$$\|u\|_{W^{2,2}(\Omega')} \leq C(\|a^{ij}\|_{\mathcal{C}^{0,1}(\Omega)}, \|b\|_{\mathcal{C}^0(\Omega)}, \|c\|_{\mathcal{C}^0(\Omega)}, \lambda, \Omega', \Omega, n) \cdot (\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}).$$

Note  $L^\infty(\Omega)$  stands for bounded functions on  $\Omega$  while  $\mathcal{C}^0(\Omega)$  are functions that are also continuous ( $\Omega$  being bounded).

*Proof.* Start with the definition of  $u$  being a solution in the weak sense,  $\forall v \in \mathcal{C}_0^1(\Omega)$ :

$$\int_{\Omega} a^{ij} D_j u D_i v = \int_{\Omega} (b^i D_i u + c - f) v.$$

and take difference quotients, that is replace  $v$  with  $\Delta^{-h} v$ .

$$\int_{\Omega} a^{ij} D_j u D_i (\Delta^{-h} v) = \int_{\Omega} (b^i D_i u + c - f) (\Delta^{-h} v).$$

Taking  $-h$  is a technicality that will unravel its reason later on, and we really mean  $\Delta_k^{-h} v$  for some  $k \in \{1, \dots, n\}$  and then eventually repeat the computation for all  $k$  in that range. This will become clear as well. Finally our goal will be to use the Chain Rule and move the difference quotient operator onto  $u$  under the integral sign and get uniform bounds on  $\Delta^h D u$  and in this way get a priori  $W^{2,2}(\Omega)$  estimates.

The Chain Rule gives

$$\begin{aligned} \Delta^h (a^{ij} D_j u) &= \\ \frac{1}{h} (a^{ij} u(x + h \cdot \mathbf{e}_k) D_j u(x + h \cdot \mathbf{e}_k) - \{a^{ij}(x) - a^{ij}(x + h \cdot \mathbf{e}_k) + a^{ij}(x + h \cdot \mathbf{e}_k)\} D_j u(x)) &= \\ = a^{ij} u(x + h \cdot \mathbf{e}_k) \Delta^h D_j u - \Delta^h a^{ij} D_j u. \end{aligned}$$

And applied to our previous equation, a short calculation verifies that we can 'integrate by part' WRT  $\Delta^h$ —

$$\begin{aligned} \int_{\Omega} a^{ij} D_j u D_i (\Delta^{-h} v) &= \int_{\Omega} \Delta^h (a^{ij} D_j u) D_i v \quad \Rightarrow \\ \int_{\Omega} a^{ij} u(x + h \cdot \mathbf{e}_k) \Delta^h D_j u D_i v &= \int_{\Omega} -\Delta^h a^{ij} D_j u D_i v + \int_{\Omega} (b^i D_i u + c - f) (\Delta^{-h} v) \quad \Rightarrow \\ \left| \int_{\Omega} a^{ij} u(x + h \cdot \mathbf{e}_k) \Delta^h D_j u D_i v \right| &\leq \|\Delta^h a^{ij} D_j u\|_{L^2(\Omega)} \|D_i v\|_{L^2(\Omega)} + \\ &\quad + \|b^i D_i u + cu - f\|_{L^2(\Omega)} \|\Delta^{-h} v\|_{L^2(\Omega)}, \end{aligned}$$

where we have used the Hölder Inequality for  $p = q = 2$ . This in turn can be bounded by

$$\begin{aligned}
&\leq \|a^{ij}\|_{C^{0,1}(\Omega)} \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + \\
&\quad + \left( \|b^i\|_{L^\infty(\Omega)} \|Du\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right) \|Dv\|_{L^2(\Omega)} \\
&\leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \cdot \|Dv\|_{L^2(\Omega)}.
\end{aligned}$$

where we have used the Hölder Inequality for  $p = 1, q = \infty$ , i.e a simple bounded integration argument (e.g  $\|cu\|_{L^2(\Omega)} = \left( \int c^2 \cdot |u|^2 \right)^{\frac{1}{2}} \leq \left( \sup |c|^2 \int_\Omega |u|^2 \right)^{\frac{1}{2}}$ ), and  $\Delta^h a^{ij} \rightarrow D_k a^{ij}$  as  $a^{ij} \in C^{0,1}(\Omega)$ .

Take a cut-off function  $\eta \in C_0^1(\Omega)$ ,  $0 \leq |\eta| \leq 1$ ,  $\eta|_{\Omega'} = 1$ . We now choose a special  $v$ :  $v := \eta^2 \Delta^h u$ .

From uniform ellipticity ( $a^{ij} \zeta_i \zeta_j \geq \lambda |\zeta|^2$ )

$$\lambda \int_\Omega |\eta D \Delta^h u|^2 \leq \int_\Omega \eta^2 a^{ij}(x + h \cdot \mathbf{e}_k) D_i \Delta^h u D_j \Delta^h u.$$

Now

$$D_i v = 2\eta D_i \eta \Delta^h u + \eta^2 D_i \Delta^h u$$

which we substitute into our previous inequality,

$$\begin{aligned}
\int_\Omega \eta^2 a^{ij}(x + h \cdot \mathbf{e}_k) D_j \Delta^h u D_j \Delta^h u &\leq \int_\Omega a^{ij}(x + h \cdot \mathbf{e}_k) D_j \Delta^h u \cdot (D_i v - 2\eta D_i \eta \Delta^h u) \\
&\leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \|Dv\|_{L^2(\Omega)} + \\
&\quad + C' \|(D \Delta^h u) \eta\|_{L^2(\Omega)} \|D \eta \Delta^h u\|_{L^2(\Omega)}
\end{aligned}$$

again by the Hölder Inequality. Now since  $\eta \leq 1$

$$\|D_i v\|_{L^2(\Omega)} \leq C'' (\|D_i \eta \Delta^h u\|_{L^2(\Omega)} + \|D \Delta^h u\|_{L^2(\Omega)}).$$

Combining all the above and again using  $\eta \leq 1$ ,

$$\begin{aligned}
\lambda \int_{\Omega}' |\eta \mathbf{D} \Delta^h u|^2 &\leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \cdot C''(\|\mathbf{D} \eta \Delta^h u\|_{L^2(\Omega')} + \|\mathbf{D} \Delta^h u\|_{L^2(\Omega')}) \\
&\quad + C'(\|\mathbf{D} \Delta^h u\|_{L^2(\Omega')}) \|\mathbf{D} \eta \Delta^h u\|_{L^2(\Omega')} \\
&\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|\mathbf{D} \eta \Delta^h u\|_{L^2(\Omega')}) \cdot \|\mathbf{D} \Delta^h u\|_{L^2(\Omega')} \\
&\quad + c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \cdot \|\mathbf{D} \eta \Delta^h u\|_{L^2(\Omega')}.
\end{aligned}$$

Using the AM-GM Inequality  $ab = \sqrt{\frac{1}{\epsilon} a^2 \cdot \epsilon b^2} \leq \frac{1}{2} \left( \frac{1}{\epsilon} a^2 + \epsilon b^2 \right)$  for the first term and the inequality

$$(a+b)c \leq \frac{1}{2}(a+b+c)^2 \quad \text{for the second}$$

$$\begin{aligned}
\lambda \int_{\Omega}' |\eta \mathbf{D} \Delta^h u|^2 &\leq \frac{1}{\epsilon} c^2 (\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|\mathbf{D} \eta \Delta^h u\|_{L^2(\Omega')})^2 + \epsilon \|\mathbf{D} \Delta^h u\|_{L^2(\Omega')}^2 \\
&\quad + c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|\mathbf{D} \eta \Delta^h u\|_{L^2(\Omega')})^2.
\end{aligned}$$

Choose any  $0 < \epsilon < \lambda/2$ . Then subtract the second term on the first line of the RHS from the LHS to get

$$\begin{aligned}
\|\eta \mathbf{D} \Delta^h u\|_{L^2(\Omega')}^2 &\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|\mathbf{D} \eta \Delta^h u\|_{L^2(\Omega')})^2 \Rightarrow \\
\|\eta \mathbf{D} \Delta^h u\|_{L^2(\Omega')} &\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|\mathbf{D} \eta \Delta^h u\|_{L^2(\Omega')}) \\
&\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \sup_{\Omega} |\mathbf{D} \eta| \cdot \|\Delta^h u\|_{L^2(\Omega')}) \\
&\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \cdot (1 + \sup_{\Omega} |\mathbf{D} \eta|),
\end{aligned}$$

since  $\|\Delta^h u\|_{L^2(\Omega)} \leq \|\mathbf{D} u\|_{L^2(\Omega)} \leq \|u\|_{W^{1,2}(\Omega)} \leq \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}$  where we have applied the first Lemma to  $u \in W^{1,2}(\Omega)$ . Now we are done as we can choose  $\eta$  such that first  $\eta|_{\Omega'} = 1$  (for the LHS !) and second  $|\mathbf{D} \eta| \leq \text{dist}(\Omega', \partial \Omega)$  (for the RHS ) and so

$$\|1 \cdot \mathbf{D} \Delta^h u\|_{L^2(\Omega')}^2 \leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}),$$

independently of  $h$ . So by our second Lemma the uniform boundedness of the difference quotients of  $Du$  in  $L^2(\Omega')$  implies  $Du \in W^{1,2}(\Omega') \Rightarrow u \in W^{2,2}(\Omega')$  and we have the desired estimate for its  $W^{2,2}(\Omega')$  norm by the above inequality combined with the Lemma. ■

Now that  $u \in W^{2,2}(\Omega')$  then the our original equation holds in the *usual* sense

$$Lu = a^{ij}D_i D_j u + D_i a^{ij}D_j u + b^i D_i u + c \cdot u = f,$$

a.e !